Bi-Hamiltonian structure of a generalized super Korteweg-de Vries equation

1993 J. Phys. A: Math. Gen. 26 L1239

(http://iopscience.iop.org/0305-4470/26/23/009)

J. Phys. A: Math. Gen. 26 (1993) L1239–L1242. Printed in the UK

LETTER TO THE EDITOR

Bi-Hamiltonian structure of a generalized super Korteweg–de Vries equation

Q P Liu

Institute of Theoretical Physics, Academia Sinica, PO Box 2735, Beijing 100080, People's Republic of China

Received 3 August 1993, in final form 7 October 1993

Abstract. The bi-Hamiltonian structure is constructed for a super extension of the KdV equation. The related Miura map is recovered by a proper gauge transformation and a new spectral problem is presented for the modified system.

In the last few years, more and more interest has been shown in the super extension of the classical integrable systems [5–6, 8–11]. The explanation for this may be that: (a) there exists a close relation between the Hamiltonian structures of a super system and super conformal algebras [2]; (b) the problem is important in its own right since integrable extensions of integrable systems is a non-trivial problem.

For the famous KdV equation, two kinds of extensions are well known: one is the so-called supersymmetric KdV equation derived by Manin and Radul [8]; the other is Kupershmidt's version [5], which is not invariant with respect to space supersymmetric transformation as Mathieu classified [9]. Interestingly, these two models can be obtained from $N=1$ super-conformal algebra [2]. Very recently, Oevel and Popowicz [11] constructed the bi-Hamiltonian structure for the Manin–Radul supersymmtric KdV equation which was not believed to exist before.

Apart from the versions of Kupershmidt and Manin–Radul, some other super extensions of the KdV equation are introduced. In particular, two vector-type sKdV systems are studied by Kupershmidt [6]. We are concerned with one of these models in this letter. The missing second Hamiltonian operator, its non-existence was claimed except in some very special cases [6], is derived explicitly. We will see that this operator is non-local. We should stress that a second Hamiltonian structure is important in view of the interaction between W-algebra and the integrable systems besides the integrable system itself. After doing this, we recover the Miura map by a proper gauge transformation. This is interesting since Kupershmidt stated that the origin of his Miura map is missing. Our approach provides a kind of origin. Furthermore, a spectral problem appeared naturally for the modified system as a by-product of this method. The letter is concluded with a discussion of some open problems.

The spectral problem we are interested in is one of the Lax operators proposed by Kupershmidt [6], to obtain some generalizations of the Korteweg–de Vries (KdV) equation. In the cited paper, Kupershmidt derived the first non-trivial flow and the 'first' Hamiltonian structure. He further claimed that the second Hamiltonian struc-

0305-4470/93/231239+04 $07.50

ture does *not* exist in general. Next, we calculate the missing second Hamiltonian operator which is proved to be a integra-differential operator.

The spectral problem reads as:

$$L=-\partial^2+u+\omega^t\partial^{-1}\omega+q\partial^{-1}r-r^t\partial^{-1}q \tag{1}$$

where superscript t denotes the transpose of a vector or matrix and we follow the usual convention. The Greek letters denote the odd (fermiomic) variables and the Latin letters the even variables with the exception that ϕ or ψ are preserved for the corresponding wave functions.

Now we rewrite the spectral problem in the following matrix form:

$$\Phi_x=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ u+\lambda & 0 & q^t & -r^t & \omega^t \\ r & 0 & 0 & 0 & 0 \\ q & 0 & 0 & 0 & 0 \\ \omega & 0 & 0 & 0 & 0 \end{bmatrix}\Phi \tag{2}$$

where $\Phi=(\phi, \phi_1, \phi_2^t, \phi_3^t, \psi^t)^t$. In order to derive the related hierarchy, we adjoint to (2) the time evolution of the wave function Φ:

$$\Phi_t\begin{bmatrix} a_1 & a_2 & A_1^t & A_2^t & \beta_1^t \\ b_1 & b_2 & B_1^t & B_2^t & \beta_2^t \\ C_1 & C_2 & M & N & \Lambda \\ D_1 & D_2 & O & P & \Delta \\ \alpha_1 & \alpha_2 & \Xi & \Pi & T \end{bmatrix}\Phi \tag{3}$$

where M, N, O, P, T, Λ, Δ, Ξ and Π are $n\times n$ matrices. Now, we calculate the zero curvature representation: $U_t-V_x+[U,V]=0$. Doing this, we obtain the following equations:

$$u_t-b_{1x}+(\lambda+u)(a_1-b_2)+q^tC_1-r^tD_1+\omega^t\alpha_1-B_1^tr-B_2^tq-\beta_2^t\omega=0 \tag{4}$$

$$q_t^t-B_{1x}^t+(\lambda+u)A_1^t+q^tM-rO+\omega^t\Pi-b_2q^t=0 \tag{5}$$

$$q_t-D_{1x}+a_1q-(\lambda+u)D_2-Or-Pq-\Delta\omega=0 \tag{6}$$

$$-r_t^t-B_{2x}^t+(\lambda+u)A_2^t+qN-r^tP+\omega^t\Pi+b_2r^t=0 \tag{7}$$

$$r_t-C_{1x}+a_1r-(\lambda+u)C_2-Mr-Nq-\Lambda\omega=0 \tag{8}$$

$$\omega_t^t-\beta_{2x}^t+(\lambda+u)\beta_1^t+q^t\Lambda-r^t\Delta+\omega^tT-b_2\omega^t-p \tag{9}$$

$$\omega_t-\alpha_{1x}+a_1\omega-(\lambda+u)\alpha_2-\Xi r-\Pi q-T\omega=0 \tag{10}$$

$$-a_{1x}+b_1-(\lambda-+u)a_2-A_1^tr-A_2^tq-\beta_1^t\omega=0 \tag{11}$$

$$-b_{2x}+(\lambda+u)a_2+q^tC_2-rD_2+\omega\alpha_2-b_1=0 \qquad -a_{2x}+b_2-a_1=0 \tag{12}$$

$$-A_{1x}+B_1-a_2q=0 \qquad -A_{2x}+B_2+a_2r=0 \qquad -\beta_{1x}+\beta_2-a_2\omega=0 \tag{13}$$

$$-C_{2x}+a_2r-C_1=0 \qquad -M_x+rA_1^t-C_2q^t=0 \qquad -N_x+rA_2^t+C_2r^t=0 \tag{14}$$

$$-\Lambda_x+r\beta_1^t-C_2\omega^t=0 \qquad -D_{2x}+a_2q-D_1=0 \qquad -O_x+qA_1^t-D_2q^t=0 \tag{15}$$

$$-P_x+qA_2^t+D_2r^t=0 \qquad -\Delta_x+q\beta_1^t-D_2\omega^t=0 \qquad -\alpha_{2x}+a_2\omega-\alpha_1=0 \tag{16}$$

$$-\Xi_x+\omega A_1^t-\alpha_2r^t=0 \qquad -\Pi_x+\omega A_2^t+\alpha_2r^t=0 \qquad -T_x+\omega\beta_1^t-\alpha_2\omega^t=0. \tag{17}$$

We need to solve the above equations so that we may obtain a hierarchy of evolution equations and its Hamiltonian structures. To this end, we solve the equations (11)–(17) and substitute them to the equations (4)–(10). After some manipulation, we have the following nice formulae:

$$\begin{bmatrix} u \\ q \\ r \\ \omega \end{bmatrix}_t = \begin{bmatrix} J_{11} & \frac{1}{2}\partial q^t + q^t\partial & \frac{1}{2}\partial r^t + r^t\partial & \frac{1}{2}\partial\omega^t + \omega^t\partial \\ \frac{1}{2}q\partial + \partial q & J_{22} & J_{23} & J_{24} \\ \frac{1}{2}r\partial + \partial r & J_{32} & J_{33} & J_{34} \\ \frac{1}{2}\omega\partial + \partial\omega & J_{42} & J_{43} & J_{44} \end{bmatrix} \begin{bmatrix} a_2 \\ 2A_2 \\ 2A_2 \\ 2\alpha_2 \end{bmatrix} \equiv (B_1 - \lambda B_0)K \tag{18}$$

where

$$J_{11} = -\tfrac{1}{2}\partial^3 + \partial(\lambda + u) + (\lambda + u)\partial \tag{19}$$

$$J_{22} = J[q, q] \equiv \tfrac{1}{2} \begin{bmatrix} q_1\partial^{-1}q_1 & \cdots & q_n\partial^{-1}q_1 \\ \vdots & \ddots & \vdots \\ q_1\partial^{-1}q_n & \cdots & q_n\partial^{-1}q_n \end{bmatrix} \tag{20}$$

$$J_{23} = \tfrac{1}{2}(\partial^2 - \lambda - u - q^t\partial^{-1}r + r^t\partial^{-1}q - \omega\partial^{-1}\omega)I + J[r, qq] \qquad J_{24} = J[\omega, q] \tag{21}$$

$$J_{32} = \tfrac{1}{2}(-\partial^2 + \lambda + u - r^t\partial^{-1}q + q^t\partial^{-1}r + \omega^t\partial^{-1}\omega)I + J[q, r] \tag{22}$$

$$J_{33} = J[r, r] \qquad J_{34} = J[\omega, r] \qquad J_{42} = J[q, \omega] \qquad J_{43} = J[r, \omega] \tag{23}$$

$$J_{44} = \tfrac{1}{2}(-\partial^2 + \lambda + u + q^t\partial^{-1}r - r^t\partial^{-1}q + \omega^t\partial^{-1}\omega)I - J[\omega, \omega] \tag{24}$$

where I denotes $n \times n$ identity matrix. It is easy to see that B_0 is just the first Hamiltonian operator given in [6]. The second Hamiltonian operator is B_1 in (18). There exist two commonly used methods to prove the Hamiltonian property for a given operator: direct calculation and Miura map approach. In the present case, the direct calculation is the only method. Indeed, it can be proved that the operator B_1 is a Hamiltonian operator. We notice that this complicated operator B_1 is reduced to the second Hamiltonian structure for the Kupershmidt sKdV when ω is a scalar and q, r vanish [5].

The Hamiltonians can be generated by considering a Riccati type equations. In fact, introducing projection coordinates, we have the following equations from the spectral problem (2):

$$y_x + y^2 = \lambda + u + q^t z_1 - r^t z_2 + \omega^t \xi \tag{25}$$

$$z_{1x} = r - z_{1y} \qquad z_{2x} = q - z_2 y \qquad \xi_x = \omega - \xi y \tag{26}$$

we seek the solutions of the form: $y = y_{-1}\zeta + y_0 + \Sigma_{i=0}^{\infty} y_i \zeta^{-i}$, $\lambda = \zeta^2$, etc. The first few solutions are:

$$y_{-1} = 1 \qquad y_1 = \tfrac{1}{2}u \qquad y_3 = -\tfrac{1}{8}u^2 + \tfrac{1}{2}(r^t q_x - q^t r_x - \omega^t \omega_x). \tag{27}$$

The related flows can now be calculated easily. In fact, the first non-trivial flow is:

$$u_t = B_1(\delta y_3) \tag{28}$$

where vecu $= (u, q^t, r^t\, \omega^t)^t$. This system is the one presented in [6] apart from some scaling of the field variables.

Now, we come to the point where we consider modifying the problem. In [6], a Miura map is presented without any indication of the methods to derive it. In fact, Kupershmidt stated that '... the Miura map does exist although its origin remains a mystery'. Here we provide an explanation. More presicely, we will derive the Miura map by a gauge transformation. Interestingly, we are provided with a spectral problem for the modified equations which was not known before.

The gauge matrix we choose is:

$$G=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -v & 1 & -a^t & b^t & -\xi^t \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix} \tag{29}$$

then an easy calculation shows that we have the following Miura type map:

$$u=v_x+v^2+a^tb_x-a_x^tb+\xi^t\xi_x \tag{30}$$

$$q=av+a_x \qquad r=bv+b_x \qquad \omega=v\gamma+\gamma_x. \tag{31}$$

We see that the map (30)–(31) is just the one which appeared in [6]. Now, it is easy to calculate the modified flow of (25) under the map (30)–(31), which is just the one which appeared in [6] apart from a scaling.

The spectral problem now is:

$$\Psi_x=\begin{bmatrix} v & 1 & a^t & b^t & \gamma^t \\ \lambda & -v & 0 & 0 & 0 \\ b_x+vb & 0 & 0 & 0 & 0 \\ a_x+va & 0 & 0 & 0 & 0 \\ \gamma_x+v\gamma & 0 & 0 & 0 & 0 \end{bmatrix}\Psi. \tag{32}$$

The modified hierarchy is claimed to be a non-Hamiltonian system [6]. However, my conjecture is that the modified system is a Hamiltonian system and the Miura type map (30)–(31) will be a Hamiltonian map. The evidence for this expectation is that the original system (28) is now a bi-Hamiltonian system, as shown above.

We conclude this note with the remark: it is pointed out in [3] (see also [7]) that the spectral $L=\partial^2+u+q\partial^{-1}r$ is related with so called W_3^2 algebra, it is interesting to see if the second Hamiltonian structure (18) leads to a new type W algebra.

I should like to thank Boris Kupershmidt for the interesting correspondence. It is my pleasure to thank the referee for comments.

References

[1] Antonowicz M and Fordy A P 1989 *Commun. Math. Phys.* **124** 465
[2] Chainchan M and Kulish P P 1987 *Phys. Lett.* **183B** 169
Mathieu P 1988 *Phys. Lett.* **203B** 287
Kupershmidt B A 1985 *Phys. Lett.* **109A** 168
[3] ven Driel P 1992 *Phys. Lett.* **274B** 179
[4] Fordy A P (ed) 1990 *Soliton Theory: A Survey of Results* (MUP)
[5] Kupershmidt B A 1984 *Phys. Lett.* **102A** 213
[6] —— 1984 *J. Phys. A: Math. Gen.* **17** L869
[7] Liu Q P and Xiong C S ASITP-93-48.
[8] Manin Yu I and Radul A O 1985 *Commun. Math. Phys.* **98** 65
[9] Mathieu P 1988 *J. Math. Phys.* **29** 2499
[10] —— 1990 *Integrable and superintegrable systems* ed B A Kupershmidt (World Scientific) pp 352–88
[11] Oevel W and Popowicz Z 1991 *Commun. Math. Phys.* **139** 441